Hyperbolic 4g-gons and Fuchsian representations(Analysis of Discrete Groups)

OKAI, Takayuki

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Takayuki OKAI (岡井 孝行)

This article is an expository summary (with Figures) of [O3].

Abstract. For any marked closed Riemann surface S with genus $g \geq 2$, we can read a corresponding Fuchsian representation from its fundamental domain of hyperbolic $4g$-gon, whose boundary consists of geodesic arcs representing generators of $\pi_1(S)$ with certain base point. Also, explicitly given is a conjugate transformation which moves such fundamental $4g$-gon to a standard position. Consequently several applications to hyperbolic geometry on S are obtained.

§0. Primitive questions

As is well-known, the hyperbolic regular $4g$-gon ($g \geq 2$) in the Poincaré disk, with all the angles equal to $\pi/2g$, gives rise to a marked closed Riemann surface of genus g, whose marking is determined by the geodesic arcs in the boundary of the original $4g$-gon. This marked Riemann surface is also characterized as the quotient of the Poincaré disk by the image of a faithful, discrete and "orientation preserving" $PSU(1,1)$-representation (we call this *"Fuchsian" representation*) of the genus g surface group.

Questions. (1) How can we describe the Fuchsian representation (up to conjugacy) for the hyperbolic regular $4g$-gon?
(2) How is the "positioning in the Riemann surface" of the base point which corresponds to the vertices of the above $4g$-gon?

[Figure 1]

§1. Marked fundamental 4g-gon and its Fuchsian representations

Let Σ_g be a closed oriented surface of genus $g \geq 2$, and fix a point $p \in \Sigma_g$. Take any hyperbolic metric h on Σ_g. Then for any $\gamma \in \pi_1(\Sigma_g, p)$, there is a unique (not always simple) geodesic arc from p to p, representing γ. Notice that this geodesic arc has a singularity at p in general. Choose a generator system $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ of $\pi_1(\Sigma_g, p)$ with the relation $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$. Suppose that for these $\alpha_1, \cdots, \beta_g$, the corresponding geodesic arc representatives are all simple and have intersections only at p. Then cutting (Σ_g, h) along such simple geodesic arcs

$$(*) \qquad \alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \cdots, \alpha_g, \beta_g, \alpha_g^{-1}, \beta_g^{-1},$$

we obtain a hyperbolic $4g$-gon with boundary corresponding to $(*)$. Hereafter we will assume that our generator systems of $\pi_1(\Sigma_g, p)$ are chosen so that the order of $(*)$ gives the clockwise orientation for the boundary.

Definition. Let $l = (l_i) \in (R_+)^g$, $\tilde{l} = (\tilde{l}_i) \in (R_+)^g$ and $\theta = (\theta_j) \in (0, 2\pi)^{4g}$. *A marked fundamental* $4g$*-gon* $X(l, \tilde{l}; \theta)$ is a hyperbolic geodesic $4g$-gon in the Poincaré disk with the clockwise namings $(*)$ of its sides, having the following properties:
(i) length of α_i = length of $\alpha_i^{-1} = l_i$, length of β_i = length of $\beta_i^{-1} = \tilde{l}_i$ $(i = 1, \cdots, g)$.
(ii) angle between α_1 and $\beta_1 = \theta_1$, angle between β_1 and $\alpha_1^{-1} = \theta_2, \cdots$, angle between β_g^{-1} and $\alpha_1 = \theta_{4g}$ (clockwise order).
(iii) $\sum_{j=1}^{4g} \theta_j = 2\pi$.

Remarks. (1) From any marked fundamental $4g$-gon, we have naturally a genus g Riemann surface with marking $(\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g)$; topologically we will regard all these marked surfaces as those with *the same marking* $(\alpha_1, \cdots, \beta_g)$. Moreover $\alpha_1, \cdots, \beta_g$ are specified as elements of $\pi_1(\Sigma_g, p)$ for the point p corresponding to the vertices of the $4g$-gon.
(2) For any marked Riemann surface of genus g, due to L. Keen [K], there are choices of base point p_0 and inner-automorphism of $\pi_1(\Sigma_g, p_0)$, so that we can construct a strictly convex marked fundamental $4g$-gon whose boundary gives the fixed genegators $\alpha_1, \cdots, \beta_g$ of $\pi_1(\Sigma_g, p_0)$. Actually Keen's construction is as follows: For any closed curve γ in a Riemann surface, let $\hat{\gamma}$ be the unique closed geodesic free-homotopic to γ. Take $p_0 = \hat{\alpha}_1 \cap \hat{\beta}_1$ and kill the ambiguity of inner-automorphisms of $\pi_1(\Sigma_g, p_0)$ in the marking $(\alpha_1, \cdots, \beta_g)$ by specifying the generators $\alpha_1 = \hat{\alpha}_1, \beta_1 = \hat{\beta}_1$. Then geodesic arcs from p_0 to p_0, corresponding to $\alpha_1, \cdots, \beta_g$ are shown to be all simple and having intersections only at p_0; thus we obtain a marked fundamental $4g$-gon from this.

Now we read a Fuchsian representation from the data of a marked fundamental $4g$-gon $X(l, \tilde{l}; \theta)$.

Notation. Denote $\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in PSU(1,1)$ (*i.e.* $|a|^2 - |b|^2 = 1$) by $[a; b]$. For $x \in R/2\pi Z$ and $y \in R$, let $e(x) = [e^{ix/2}; 0]$ (rotation of angle x around 0 in the Poincaré disk) and $eh(y) = [ch(y/2); sh(y/2)]$ (hyperbolic displacement of length y along the real axis).

Theorem 1. *The following gives a corresponding Fuchsian representation* ρ *for any marked fundamental* $4g$*-gon* $X(l,\tilde{l};\theta)$:

$$\rho(\alpha_i) = e(\theta_1 + \cdots + \theta_{4i-4})eh(l_i)e(\pi - (\theta_{4i-2} + \theta_{4i-1}))e(-(\theta_1 + \cdots + \theta_{4i-4})),$$
$$\rho(\beta_i) = e(\theta_1 + \cdots + \theta_{4i-1})eh(\tilde{l}_i)e(-\pi + (\theta_{4i-3} + \theta_{4i-2}))e(-(\theta_1 + \cdots + \theta_{4i-1})).$$

Proof. First fix a position of $X(l,\tilde{l};\theta)$ in the Poincaré disk by lifting $p \in \Sigma_g$ (this is the point corresponding to the vertices of $X(l,\tilde{l};\theta)$) to the origin and also lifting $\alpha_1 \subset \Sigma_g$ (geodesic arc from p to p) to the real axis. In this situation, we will read the corresponding holonomy representation, say, ρ.

[Figure 2]

The lift $\tilde{\alpha}_i$ of α_i, starting from the origin, is described as follows:

[Figure 3]

From the direction of the real axis, rotate by the angle $\theta_1 + \cdots + \theta_{4i-4}$ and go straight by the length l_i (the reaching point will be denoted as $\rho(\alpha_i)\cdot 0$). At the point p, the angle from the incoming direction of α_i (here p is the end point) to the outgoing direction of α_i (here p is the starting point) is equal to $\pi - (\theta_{4i-2} + \theta_{4i-1})$. Thus by $\rho(\alpha_i)$, the direction of $\tilde{\alpha}_i$ at 0 is mapped to the direction of angle $\pi - (\theta_{4i-2} + \theta_{4i-1})$, measured from the direction of $\tilde{\alpha}_i$ at $\rho(\alpha_i)\cdot 0$. Notice that the above data determine the element $\rho(\alpha_i)$ of $PSU(1,1)$. Now from Figure 4, we can see that the right-hand side of the formula for α_i in the statement of Theorem 1 actually coincides with the element $\rho(\alpha_i)$.

[Figure 4]

The case for $\rho(\beta_i)$ is as well. □

Remark. Suppose that there is a hyperbolic $4g$-gon $X(l,\tilde{l};\theta)$ with the conditions (i), (ii) and (iii)$'$ $\sum_{j=1}^{4g} \theta_j = \omega$, instead of (iii). Then by a direct calculation, we have, for the ρ in Theorem 1,

$$[\rho(\alpha_1), \rho(\beta_1)] \cdots [\rho(\alpha_g), \rho(\beta_g)] e(\omega) =$$
$$\prod_{i=1}^{g} eh(l_i)e(-\pi + \theta_{4i-3})eh(\tilde{l}_i)e(-\pi + \theta_{4i-2})eh(l_i)e(-\pi + \theta_{4i-1})eh(\tilde{l}_i)e(-\pi + \theta_{4i}),$$

where $\prod_{i=1}^{g} A_i$ means $A_1 \cdots A_g$. We can see that the right-hand side is equal to $I \in PSU(1, 1)$ (cf. [O2, Lemma]), which is equivalent to the condition that $X(l, \tilde{l}; \theta)$ is a hyperbolic $4g$-gon. Thus the above $X(l, \tilde{l}; \theta)$ and ρ give rise to a developing map of a genus g hyperbolic cone manifold with one cone point p of cone angle ω.

§2. Moving a marked fundamental 4g-gon to the standard position

Suppose that we are given two marked fundamental $4g$-gons $X = X(l, \tilde{l}; \theta)$ and $X' = X(l', \tilde{l}'; \theta')$; by Theorem 1, we have the corresponding Fuchsian representations ρ and ρ'. X and X' give the same marked Riemann surface $[(\Sigma_g, h), (\alpha_1, \cdots, \beta_g)]$ (*i.e.* the same element of the genus g Teichmüller space $\mathcal{T}_g$) if and only if ρ and ρ' are conjugate to each other by an element of $PSU(1, 1)$. In this section, we will give a criterion for these. Of course, it is possible to choose more than ($6g$-6) elements in $\pi_1(\Sigma_g)$, so that the geodesic lengths of these elements give a global coordinate system for $\mathcal{T}_g$. In comparison, our method is more geometrical and direct one. We will construct conjugate transformations which move X and X' to standard positions (see below). Then applying such transformations to $\rho(\alpha_1), \cdots, \rho(\beta_g)$ and $\rho'(\alpha_1), \cdots, \rho'(\beta_g)$, we can answer whether ρ is conjugate to ρ' or not.

Definition. A marked fundamental $4g$-gon in the Poincaré disk (or, its associated Fuchsian representation ρ constructed in Theorem 1) is said to be *in the standard position* if the axes of $\rho(\alpha_1)$ and $\rho(\beta_1)$, denoted by $ax(\rho(\alpha_1))$ and $ax(\rho(\beta_1))$, satisfy that $ax(\rho(\alpha_1)) =$ the real axis and $ax(\rho(\alpha_1)) \cap ax(\rho(\beta_1)) = \{0\}$.

Remark. $ax(\rho(\alpha_1)), ax(\rho(\beta_1))$ are lifts of the closed geodesics $\widehat{\alpha}_1, \widehat{\beta}_1 \subset \Sigma_g$, respectively. These axes have a transverse intersection because there exists (see § 1, Remarks (1), (2)) a path $\epsilon \subset \Sigma_g$, from p (the point corresponding to vertices of the $4g$-gon) to $p_0 = \widehat{\alpha}_1 \cap \widehat{\beta}_1$, such that $\epsilon \widehat{\alpha}_1 \epsilon^{-1} \simeq \alpha_1$ and $\epsilon \widehat{\beta}_1 \epsilon^{-1} \simeq \beta_1$ in Σ_g (lifts of p and ϵ determine the point $ax(\rho(\alpha_1)) \cap ax(\rho(\beta_1))$).

Theorem 2.1. *For any marked fundamental $4g$-gon $X(l, \tilde{l}; \theta)$, we can explicitly give the conjugate transformation which moves its associated Fuchsian representation ρ to the standard position.*

Proof. The idea is to use the Iwasawa decomposition of $PSU(1, 1)$: Let $K = \{[e^{i\varphi}; 0]; \varphi \in$

$R/2\pi Z\}$, $N = \{[1+ir;ir];r \in R\}$ and $A = \{[ch(\lambda);sh(\lambda)];\lambda \in R\}$. Then we have $PSU(1,1) = ANK$ and we will determine the desired transformation, first for the components of N and K, and second for the component of A.

Step 1. We will determine the element $P(\rho(\alpha_1)) = nk$ ($n \in N$ and $k \in K$) such that $P(\rho(\alpha_1)) \circ \rho(\alpha_1) \circ P(\rho(\alpha_1))^{-1} = [ch(L);sh(L)]$ for some $L > 0$.

[Figure 5]

Actually we can treat with this problem in a more general setting: Given $[p_1+ip_2;q_1+iq_2] \in PSU(1,1)$ with $p_1 > 1$, we will solve the following equation for $n = [1+ir;ir] \in N$ and $k = [e^{i\varphi};0] \in K$;

$$(2.1) \qquad nk[p_1+ip_2;q_1+iq_2](nk)^{-1} = [ch(L);sh(L)].$$

By a direct calculation, we can see that (2.1) holds if and only if $p_1 = ch(L)$, $q_2\cos(2\varphi) + q_1\sin(2\varphi) = p_2$, $q_1\cos(2\varphi) - q_2\sin(2\varphi) = sh(L)$ and $r = -p_2/2sh(L)$. We look for the solution with $L > 0$, so $sh(L) = (p_1^2-1)^{1/2}$. Now let $\Psi \in R/2\pi Z$ be the angle with $\cos\Psi = q_1/(q_1^2+q_2^2)^{1/2}$ and $\sin\Psi = q_2/(q_1^2+q_2^2)^{1/2}$. (Notice that if $q_1 = q_2 = 0$, then we have $p_2 = 0$ and thus $p_1 = 1$.) Then we have $\sin(2\varphi+\Psi) = p_2/(q_1^2+q_2^2)^{1/2}$ and $\cos(2\varphi+\Psi) = (p_1^2-1)^{1/2}/(q_1^2+q_2^2)^{1/2}$. (Notice that $|p_2/(q_1^2+q_2^2)^{1/2}| \leq 1$ if and only if $p_1^2 \geq 1$.) These formulas determine $2\varphi+\Psi \in R/2\pi Z$, and thus determine $\varphi \in R/\pi Z$. In this way we can determine $r \in R$ and $\varphi \in R/\pi Z$ from (2.1). (In particular for $\rho(\gamma) = e(\psi_1)eh(s)e(\psi_2)$, let $\Phi = (\psi_1+\psi_2)/2 \in R/2\pi Z$ with $\cos\Phi > 0$ and $\Psi = (\psi_1-\psi_2)/2 \in R/2\pi Z$, so that $\Phi + \Psi = \psi_1$. Then $P(\rho(\gamma)) = [1+ir;ir][e^{i\varphi};0]$ is determined by $e^{i(2\varphi+\Psi)} = ((\cos\Phi)^2 ch(s/2)^2 - 1)^{1/2} + i\sin\Phi/th(s/2)$ and $r = -\tan(2\varphi+\Psi)/2$.)

Step 2. Because the group A consists of hyperbolic displacements along the real axis and $ax(\rho(\alpha_1))$ and $ax(\rho(\beta_1))$ intersect transversely, there exist unique elements $eh(2\lambda), eh(2\tilde{\lambda}) \in A$ such that

$$(2.2) \qquad eh(2\tilde{\lambda})P(\rho(\beta_1))(eh(2\lambda)P(\rho(\alpha_1)))^{-1}\cdot 0 = 0$$

(here $\cdot$ means a fractional linear transformation; $[a;b]\cdot z = (az+b)/(\bar{b}z+\bar{a})$).

[Figure 6]

This $eh(2\lambda) \in A$ is exactly the one what we want; $eh(2\lambda)P(\rho(\alpha_1))$ moves ρ to the standard position. To get the formula for λ, we have to solve the equation (2.2) for $\lambda = \lambda(\rho)$ and

$\tilde{\lambda} = \tilde{\lambda}(\rho)$. Write $P(\rho(\beta_1)) \circ P(\rho(\alpha_1))^{-1} = [a_1 + ia_2; b_1 + ib_2]$. Then (2.2) is equivalent to $-sh(\lambda - \tilde{\lambda})a_1 + ch(\lambda - \tilde{\lambda})b_1 = 0$ and $-sh(\lambda + \tilde{\lambda})a_2 + ch(\lambda + \tilde{\lambda})b_2 = 0$. Notice that we have $a_1 \neq 0$ and $a_2 \neq 0$; otherwise the axes $ax(\rho(\alpha_1))$ and $ax(\rho(\beta_1))$ would coincide (orientation preservingly or reversingly) with each other. Thus from $th(\lambda - \tilde{\lambda}) = b_1/a_1$ and $th(\lambda + \tilde{\lambda}) = b_2/a_2$, we can get the formula for λ: $sh(\lambda) = \{|((a_1 + b_1)(a_2 + b_2))/((a_1 - b_1)(a_2 - b_2))|^{1/4} - |((a_1 - b_1)(a_2 - b_2))/((a_1 + b_1)(a_2 + b_2))|^{1/4}\}/2$. □

Remarks. (1) In the above, $|a_1| > |b_1|$ and $|a_2| > |b_2|$ must be satisfied because (2.2) has unique solutions λ and $\tilde{\lambda}$.
(2) Step 1 and Step 2 can be automatically applied to two hyperbolic transformations H_1, H_2 with their axes having transverse intersections; we can give the explicit formula for the transformation which moves $ax(H_1)$ to the real axis and $ax(H_1) \cap ax(H_2)$ to 0.

As a summary of this section, we shall record the following

Theorem 2.2. *For two marked fundamental $4g$-gons X and X', let ρ and ρ' be their associated Fuchsian representations constructed in Theorem 1. Then ρ and ρ' are conjugate in $PSU(1, 1)$ (i.e. give the same element of T_g) if and only if $eh(2\lambda)P(\rho(\alpha_1)) \circ \rho(\gamma) \circ (eh(2\lambda) P(\rho(\alpha_1)))^{-1} = eh(2\lambda')P(\rho'(\alpha_1)) \circ \rho'(\gamma) \circ (eh(2\lambda')P(\rho'(\alpha_1)))^{-1}$ for $\gamma = \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$, where $P(\ )$ is given in Theorem 2.1, Step 1, and $\lambda = \lambda(\rho)$ and $\lambda' = \lambda(\rho')$ are given in Theorem 2.1, Step 2.* □

§3. Applications

Once we know a Fuchsian representation (Theorem 1) and the standard position (Theorem 2.1) of a marked fundamental $4g$-gon, we can investigate hyperbolic geometry of closed Riemann surfaces, in detail and in a direct way.

Proposition 3.1. *For any marked fundamental $4g$-gon $X(l, \tilde{l}; \theta)$, let $\rho : \pi_1(\Sigma_g, p) \to PSU(1, 1)$ be its Fuchsian representation given in Theorem 1 (recall that, here p is corresponding to the vertices, 0 is a lift of p and the real axis is a lift of α_1). Let $\delta \subset \Sigma_g$ be the geodesic arc from $p_0 = \hat{\alpha}_1 \cap \hat{\beta}_1$, to p such that $\delta^{-1}\hat{\alpha}_1\delta \simeq \alpha_1$ and $\delta^{-1}\hat{\beta}_1\delta \simeq \beta_1$. Then in the standard position of $X(l, \tilde{l}; \theta)$, we can write down the positioning of the lift $\tilde{\delta}$ of δ, starting from 0.*

Proof. We use the notation of Theorem 2.1. The end-point w of $\tilde{\delta}$ is given by $w = eh(2\lambda)P(\rho(\alpha_1)) \cdot 0$. Explicitly we have the following formula:

$$w = (ch(\lambda)r\sin\varphi + sh(\lambda)(\cos\varphi - r\sin\varphi) + i(ch(\lambda)r\cos\varphi - sh(\lambda)(r\cos\varphi + \sin\varphi))/$$
$$(ch(\lambda)(\cos\varphi - r\sin\varphi) + sh(\lambda)r\sin\varphi - i(ch(\lambda)(r\cos\varphi + \sin\varphi) - sh(\lambda)r\cos\varphi)). \square$$

Proposition 3.2. *For any marked fundamental $4g$-gon $X(l, \tilde{l}; \theta)$ and its associated Fuchsian representation ρ constructed in Theorem 1, let $X(l^0, \tilde{l}^0; \theta^0)$ and $\rho_0 : \pi_1(\Sigma_g, p_0) \rightarrow PSU(1,1)$ be the unique marked fundamental $4g$-gon and its associated Fuchsian representation such that $\alpha_1 = \hat{\alpha}_1, \beta_1 = \hat{\beta}_1$ $\hat{\alpha}_1 \cap \hat{\beta}_1 = \{p_0\}$ and ρ_0 is conjugate to ρ in $PSU(1,1)$. Then we can write down these "canonical" parameters $l^0, \tilde{l}^0$ and θ^0 as functions of $l, \tilde{l}$ and θ.*

Proof. By the construction of ρ in Theorem 1, ρ_0 is by itself in the standard position. Thus we have $\rho_0(\gamma) = eh(2\lambda(\rho))P(\rho(\alpha_1))\circ\rho(\gamma)\circ(eh(2\lambda(\rho))P(\rho(\alpha_1)))^{-1}$ (here $\gamma \in \pi_1(\Sigma_g, p_0)$ and $\gamma \in \pi_1(\Sigma_g, p)$ are identified by the path δ in Proposition 3.1). Let $\rho_0(\gamma) \cdot 0 = z(\gamma)$. Then l_i^0 and $\tilde{l}_i^0$ are given by $l_i^0 = d_P(0, z(\alpha_i))$ and $\tilde{l}_i^0 = d_P(0, z(\beta_i))$, where $d_P(0, z) = \log\{(1+|z|)/(1-|z|)\}$, the Poincaré metric.

[Figure 7]

Let us deduce the formula for θ_i^0, for example for θ_5^0, the angle between the sides α_2 and β_2 of $X(l^0, \tilde{l}^0; \theta^0)$. In our orientation convention, θ_5^0 is nothing but the angle from the vector $z(\alpha_2^{-1})$ to $z(\beta_2)$; thus we have $e^{i\theta_5^0} = (z(\beta_2)/|z(\beta_2)|)/(z(\alpha_2^{-1})/|z(\alpha_2^{-1})|)$. $\square$

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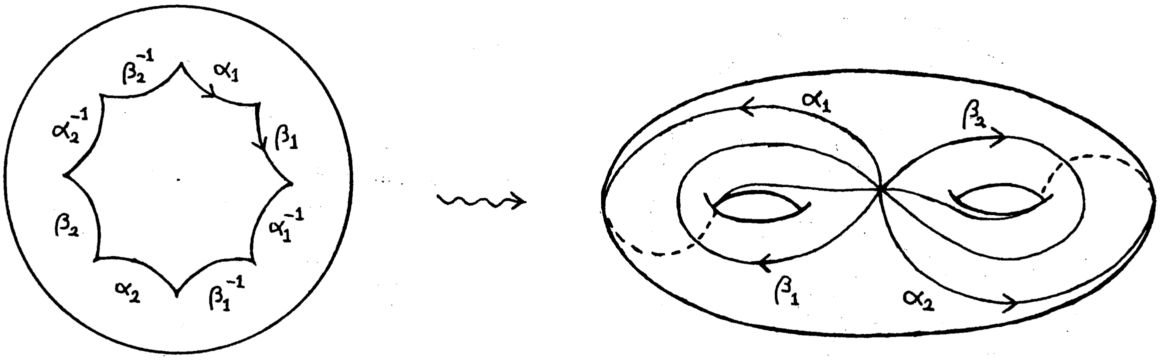

Figure 1

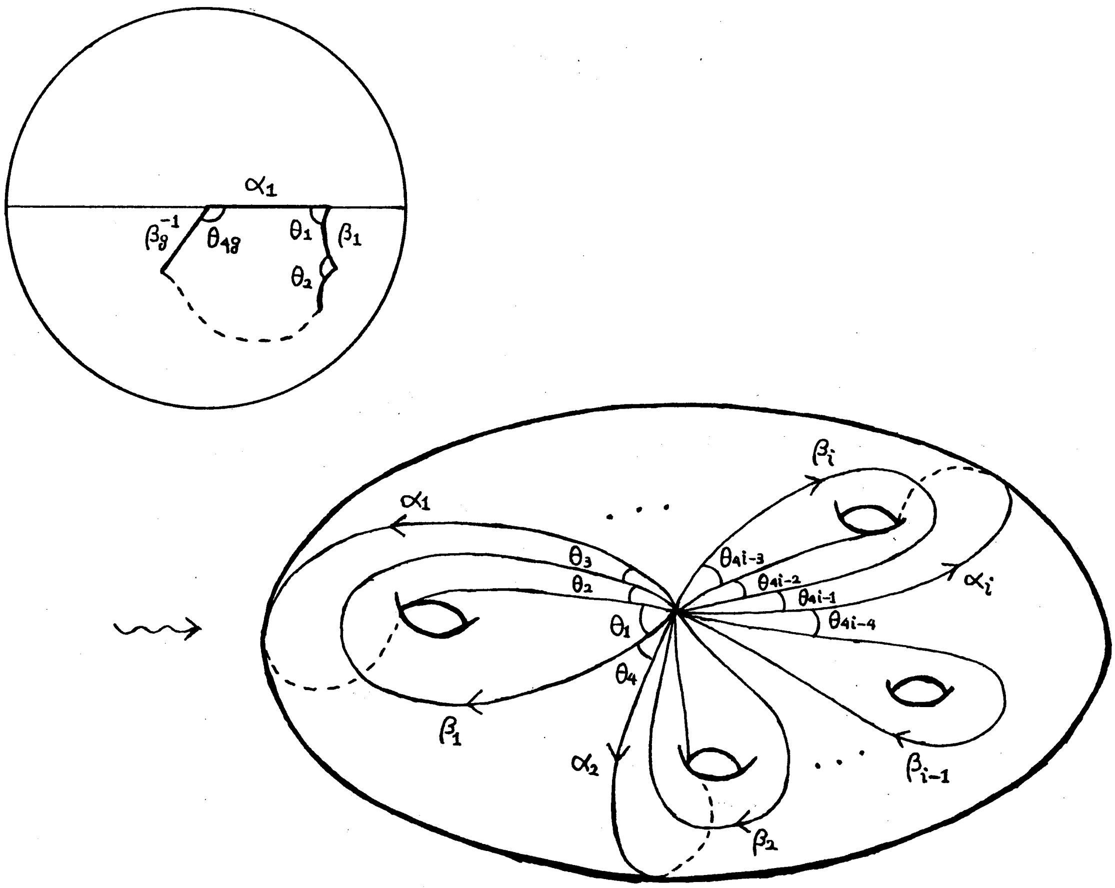

Figure 2

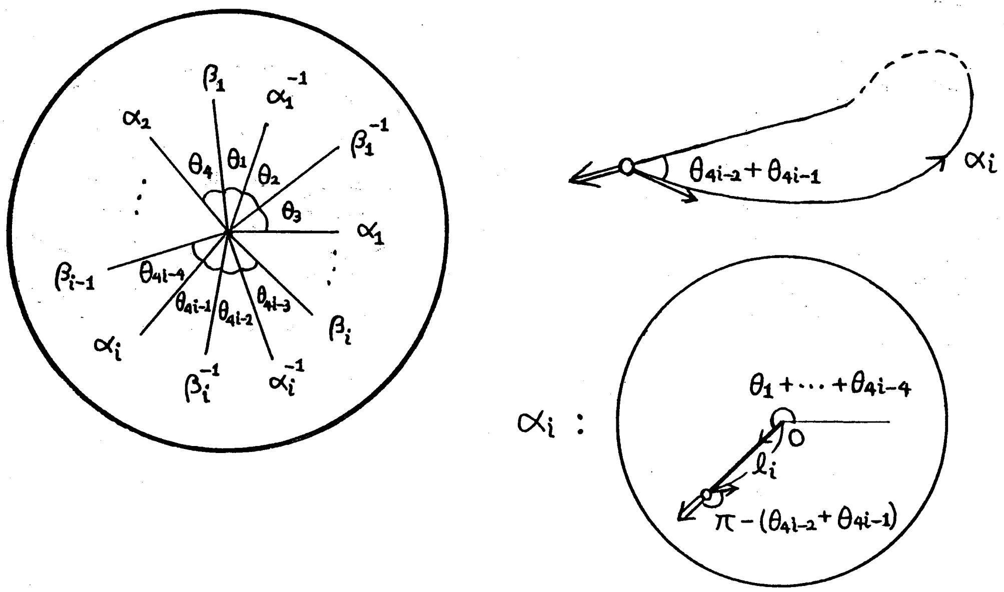

Figure 3

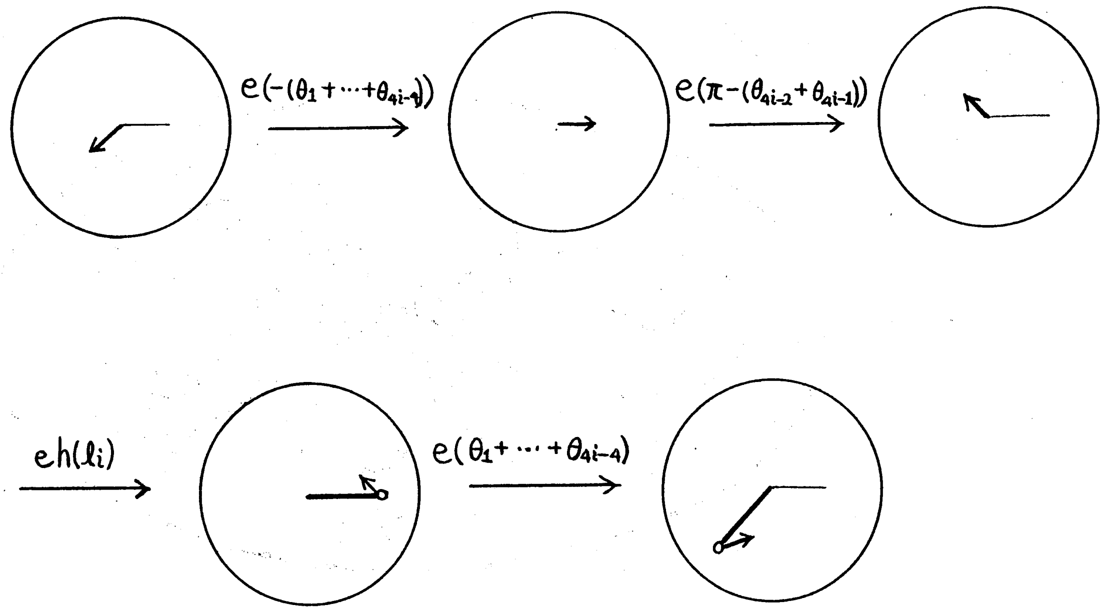

Figure 4

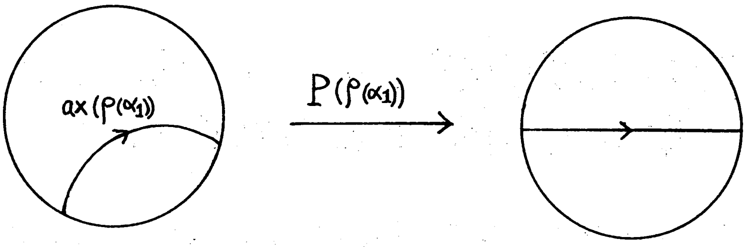

Figure 5

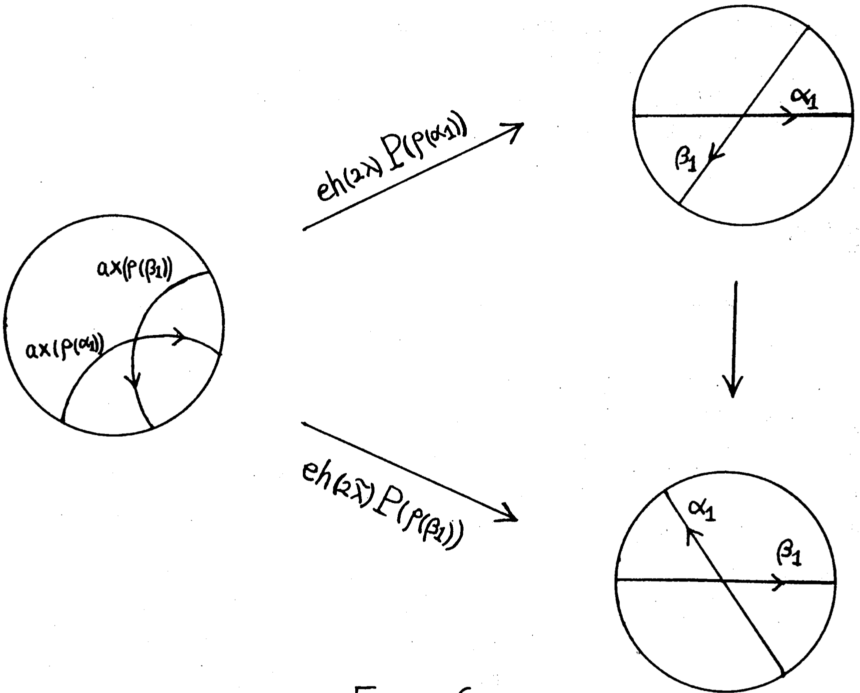

Figure 6

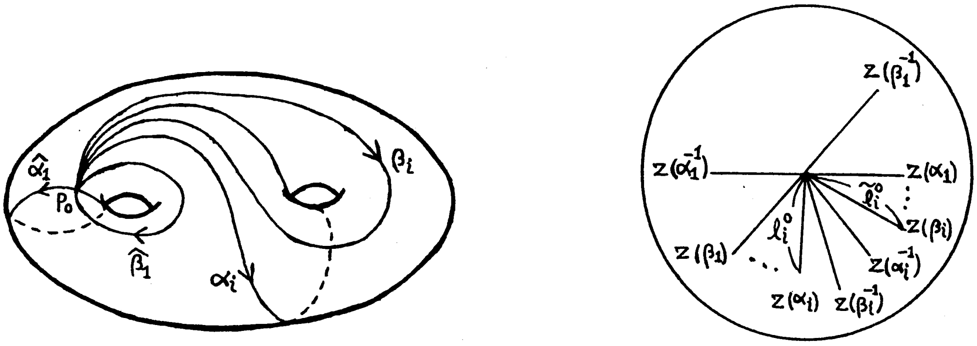

Figure 7